Modeling of Gauss elimination technique for multi-objective linear programming problem

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ABSTRACT: In many decision problems, criteria can be expressed as linear polynomial and corresponding optimization problems are multi-objective programs of linear type. Modeling of Gauss elimination technique of inequalities is derived for numerical solution of multi-objective linear programming problem by using concept of bounds. The method is quite useful because the calculations involved are simple and take least time. The method has been illustrated by a numerical example also.

Keywords Multiple criteria, Decision analysis, Elimination technique, Multi-objective linear programming problem, Inequalities.

INTRODUCTION

The purpose of this paper is to give a brief taxonomy of an important group of programming problems which occur in certain branches of applied mathematics. A rough heading for these problems is "Modeling of Mathematical Programming Problem". Here special emphasis is given on multi-objective optimization problem. It is no accident that the principal source of such problems is multiple decision-making in which the action space is of a multi-objective nature. Iterative approaches involving interactions with decision makers have often been advocated in multiple criterion optimizations as well as in the specification of politician's preferences in macro-economic decisions involving multiple objectives.

Multi-objective linear programming problem (MOLPP) emerges naturally in decision making when several rates are to be optimized simultaneously and a compromise is sought which optimizes a weighted sum of these rates. In the light of the applications of single objective linear programming problem, objective function may represent output, input, profit, cost, capital, return on investment, productivity, liquidity, risk or time (Kornbluth and Steuer 1981) . A multitude of applications of the MOLPP can be envisioned in this way. The important applications of MOLPP can be seen in several areas such as production, transportation (marine), finance (corporate planning, bank balance sheet), engineering, statistics, water resources, health care and so forth (Steuer 1986 & Korhonen, Salo and Steuer 1997).

Traditional mathematical programming models are based on the assumption that the decision making has a single, quantifiable, objective such as maximization of profit or minimization of inefficiency or cost. However, often there are situations, where instead of posing a single objective, managers use comparative criteria in decision-making. Thus, instead of setting only one objective, multiple goals in form of linear may be set. Specification of multiple goals creates difficulties in the solution to a given problem because the objectives are usually conflicting and incommensurate.

Many researchers have proposed algorithms for solving mathematical programming problem, for example: Charnes and Cooper (1962), Martos (1975), Wolf(1985) and others. Comparative investigations of such algorithms can be found in Arsham and Kahn (1990) and Bhatt (1989). Additional information concerning especially the 'bad

points (A point x' is called a bad point if $f(x)$ tending to infinity when x tending to x') is studied by Verma, Khanna and Puri (1989).

A good amount of research is going on to improve the available elimination algorithms for solving linear programming models. Karmarker (1984) developed algorithm for linear program which is polynomially bounded. Williams (1986) showed that the method discovered by Fourier (Kohler 1973) for manipulating linear inequalities can be used to solve linear programming problem by choosing a variable for elimination, which will generate minimum number of constraints in each steps. Kanniappan and Thangavel (1998) and Bhargava and Sharma (2003) developed algorithms for solving linear programming problem and integer linear programming problem respectively by eliminating algorithm. Later Jain and Mangal (2008) developed Gauss elimination technique for fractional programming problem and also for integer solution.

For the first time an elimination algorithm is modeled for numerical solution of multi-objective linear programming problem by using Williams (1986), Kanniappan and Thangavel (1998), Jain (2009) work's in modified form for MOLPP by using Fourier method. Our aim is to reduce the computing time of the optimization process of the proposed problem in which multi-objective function treated as constraints in nature. These constraints are encountered in transportation, flow and network models. It is assumed that the set of the feasible solutions is a convex polyhedral with a finite number of extreme points and that the denominator of the quotient function is non-zero on the constraint set.

In the following section, we describe in brief how Gauss elimination technique is applied on inequalities. In the proposed method, first we convert multi-objective function of MOLPP into inequalities. Using updated form of Gauss elimination technique to solve these systems of inequalities of MOLPP and finally to find solution. This has been verified by a numerical example and concluding remarks are given at the end.

Gauss elimination technique for inequalities

In numerical analysis, the systems of simultaneous linear equation is solved by Gauss elimination technique with the help of elimination of variables one by one and finally reduce to upper triangular system of equations, which can be solved by back substitution. Equation gives only one solution while inequality gives possibility of many solutions in bounded / region form, out of which we select maximum or minimum value according to the problem to optimize. This is the main theme to apply Gauss elimination technique for inequality in place of equation. It can easily verify that max. /mini. value of linear variables gives, max. /mini. value of objective function of $\sum C_i X_i$ form where all C_i are positive. If some C_i are not positive, then we take mini. /max. value of corresponding linear variables so it gives max. /mini. value of objective function.

Here we apply Gauss elimination technique for a system of inequalities of the same sign i.e., either less than equal to ($\leq$) or greater than equal to ($\geq$) in nature. Variables are eliminated by combining inequalities in such a way that the inequalities and variables are reduced one by one in every iteration i.e., one variable and one inequality reduce in one iteration so that at last there remains only one inequality with one variable. This last inequality gives value of last variable in bounded form and finally the value of last variable is taken at its maximum or minimum according to objective function of MOLPP. Finally, we get value of other variables by back substitution of value of the last variable.

For the sake of clarity and simplicity we consider the system of inequalities of n variables and m inequalities:

$$a_{11}x_1 + ... + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + ... + a_{2n}x_n \leq b_2$$
$$... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ...$$
$$... \quad ... \quad ... \quad ... \quad ... \quad ... \quad ...$$
$$a_{m1}x_1 + ... + a_{mn}x_n \leq b_m$$

First of all to eliminate the first variable say x_1 multiply the first row by $\frac{a_{21}}{a_{11}}, \frac{a_{31}}{a_{11}}, ..., \frac{a_{m1}}{a_{11}}$ respectively and then subtract them from second, third and so on up to the mth row respectively. In this stage, we have assumed that $a_{11} \neq 0$. The first equation is called pivotal equation and $a_{11} \neq 0$ is called the first pivot. Then we get first iteration, which is

$$a_{22}^{(2)} x_2 + ... + a_{2n}^{(2)} x_n \leq b_2^{(2)}$$

$$a_{32}^{(2)} x_2 + ... + a_{3n}^{(2)} x_n \leq b_3^{(2)}$$

... ...

... ...

$$a_{m2}^{(2)} x_2 + ... + a_{mn}^{(2)} x_n \leq b_m^{(2)}$$

$$where \ a_{mn}^{(2)} = a_{mn} - {a_{m1} a_{1n}}/{a_{11}}$$

$$and \ \ b_k^{(2)} = b_k - b_1 \frac{a_{k1}}{a_{11}}$$

After first iteration, we get *(n-1)* variables and *(m-1)* inequalities. Repeating this process or after *(n-1)* iterations we are left with one variable only. Finally, we get value of other variables by back substitution of the value of last variable. There may be some redundant constraints present in the system.

Problem formulation for Gauss elimination algorithm

Here we consider the multi-objective linear programming problem as:

$$Max Z = \{ a_1 x + \alpha_1 = z_1, \ a_2 x + \alpha_2 = z_2, ..., a_n x + \alpha_n = z_n \}$$

$$Subject \ to, \ A \, x \leq b$$

$$and \qquad x \geq 0$$

. . . (3.1)

To apply Gauss elimination technique on MOLPP, we formulate this MOLPP by taking objective function as constraint and all constraints of the same sign of inequalities. Thus reduced form of MOLPP for Gauss elimination technique is as follows:

$$Max Z = Max \{ z_1, z_2, ..., z_n \}$$

$$(a_1 x + \alpha_1) - z_1 \leq 0$$

$$(a_2 x + \alpha_2) - z_2 \leq 0$$

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.

.

$$(a_n x + \alpha_n) - z_n \leq 0$$

$$A \, x \leq b$$

$$- x \leq 0$$

. . . (3.2)

Now combining inequalities in such a way that the inequalities and variables are reduced one by one in each iteration eliminates the variables. If at any stage we get an absurd inequality like 0 < d where d is a negative number then we conclude that the given MOLPP has infeasible solution otherwise MOLPP has feasible solution. To explain the whole procedure we consider a numerical example of MOLPP which having optimal/feasible solution in the next section.

Numerical example

Consider the following MOLPP

$$Max \ z = \{3x + 2y = z_1, \ x + 3y = z_2\}$$
$$subject \ to, \qquad x + y \leq 4$$
$$x - y \leq 2$$
$$and \qquad x, y \geq 0$$

Making standard form by treating objective function as constraint and all inequalities of same sign for Gauss elimination technique, we have

$$Max \ \{z_1, z_2\}$$

$$3x + 2y - z_1 \leq 0 \qquad\qquad \text{... (4.1)}$$
$$x + 3y - z_2 \leq 0 \qquad\qquad \text{... (4.2)}$$
$$x + y \leq 4 \qquad\qquad \text{... (4.3)}$$
$$x - y \leq 2 \qquad\qquad \text{... (4.4)}$$
$$-x \leq 0 \qquad\qquad \text{... (4.5)}$$
$$-y \leq 0. \qquad\qquad \text{... (4.6)}$$

In the first stage of Gauss elimination technique with the help of above equations the variable x is selected for eliminating. Eliminate x by Gauss elimination technique. We have,

$$Max \ \{z_1, z_2\}$$

$$-y - z_1 \leq -12 \qquad\qquad \text{... (4.7)}$$
$$2y - z_2 \leq -4 \qquad\qquad \text{... (4.8)}$$
$$y \leq 1 \qquad\qquad \text{... (4.9)}$$
$$-y \leq 2 \qquad\qquad \text{...(4.10)}$$
$$-y \leq 0 \ . \qquad\qquad \text{.. (4.11)}$$

Now after eliminating x, the only left variable $j = 2$ i.e. y, from the above inequalities eliminate y and the reduced inequalities are

Max $\{z_1, z_2\}$

$$-z_1 \leq -11 \qquad\qquad \text{...(4.12)}$$
$$-z_2 \leq -6 \qquad\qquad \text{...(4.13)}$$
$$0 \leq 4 \qquad\qquad \text{...(4.14)} \qquad\qquad 0 \leq 2$$
$$\text{...(4.15)}$$

In this step, all the variables are eliminated except the variables z_1, z_2 and the least upper bound for the variables $z_1 = 11$ and $z_2 = 6$ obtained from the inequalities (4.12) and (4.13).

Now by putting $z_1 = 11$, $z_2 = 6$ in the inequalities (4.7) to (4.11), we get different bounded values for variable y {- $y \leq -1$, $y \leq 1$, $y \leq 1$, $-y \leq 2$ and $y \geq 0$}. Out of this $y = 1$ is the only value that satisfies all the inequalities from (4.7) to (4.11) simultaneously. Therefore $y = 1$.

Now by putting $z_1 = 11$, $z_2 = 6$ and $y = 1$ in the inequalities (4.1) to (4.6), we get different bounded values for variable x { $x \leq 3$ and $x \geq 0$}. Out of this $x = 3$ is the only value that satisfies all the inequalities from (4.1) to (4.6) simultaneously. Therefore $x = 3$.

Hence the numerical solution of above MOLPP's is $x = 3$, $y = 1$ and $z_1 = 11$, $z_2 = 6$.

CONCULSION

The technique discussed in this paper provides an interactive approach in which the decision maker can search for an acceptable solution of the multi-objective optimization problem. The proposed method to solve multiobjective linear programming problem is better than many existing methods as the concept of bound is used in the iteration.

If we substitute some values to a_i , α_i in multi-objective linear programming problem (3.1), it reduces into single objective LPP. This discussion also holds in the case as given by by Kanniappan and Thangavel (1998). The same problem for integer solution was studied by Bhargava and Sharma (2003).

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